North Sydney Girls High School

2022 **HSC TRIAL EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading Time – 10 minutes
- Working Time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7 – 13)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

NAME:____________________________ **TEACHER:**____________________________

STUDENT NUMBER:

Question	1-10	11	12	13	14	Total
Mark	/10	/15	/15	/15	/15	/70

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1 Let $P(x) = x^3 - 2ax^2 + x - 1$ where $a \in \mathbb{R}$. When $P(x)$ is divided by $x+2$, the remainder is 5. What is the value of a?

A. 2

B. $-\dfrac{7}{4}$

C. $\dfrac{1}{2}$

D. -2

2 The points A and B have coordinates $(-2,3)$ and $(2,-5)$ respectively.
Which of the following is the vector $\overrightarrow{AB}$?

A. $-2\underset{\sim}{j}$

B. $4\underset{\sim}{i} - 8\underset{\sim}{j}$

C. $-4\underset{\sim}{i} + 8\underset{\sim}{j}$

D. $2\underset{\sim}{j}$

3 What is the angle between the vectors $\begin{pmatrix} -7 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

A. $\cos^{-1}(-0.8)$

B. $\cos^{-1}(-0.08)$

C. $\cos^{-1}(0.8)$

D. $\cos^{-1}(0.08)$

4 Which of the following is the derivative of $\tan^{-1}(3x)$?

A. $3\tan^{-1}3x$

B. $\dfrac{3}{1+9x^2}$

C. $\dfrac{3}{1+3x^2}$

D. $3\sec^2 3x$

5 What is the equation of the inverse of $f(x)=\dfrac{5+e^{2x}}{3}$?

A. $y=\dfrac{3}{5+e^{2x}}$

B. $y=e^{5-3x}$

C. $y=\dfrac{1}{2}\ln\left(3x-5\right)$

D. $y=\dfrac{1}{2}\ln\left(5-3x\right)$

6 Four female and four male students are to be seated around a circular table.
In how many ways can this be done if the males and females must alternate?

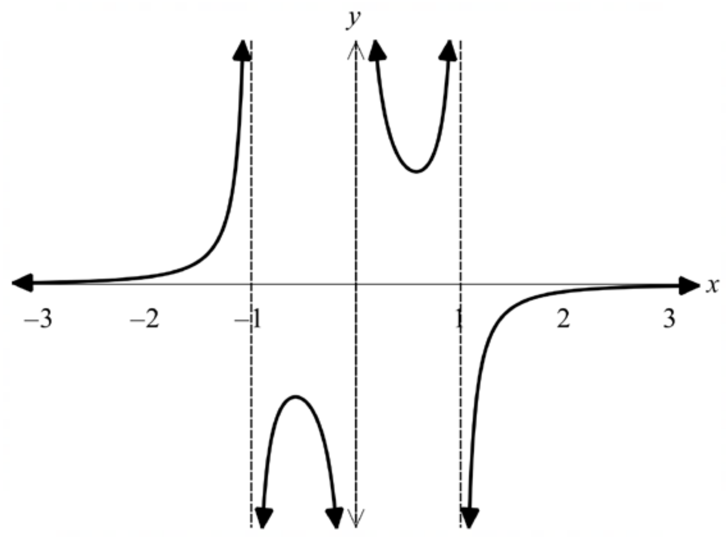
A. $4!\times 4!$

B. $3!\times 4!$

C. $3!\times 3!$

D. $2\times 3!\times 3!$

7 The graph below shows $y = \dfrac{1}{f(x)}$.



Which of the following best represents the equation of $f(x)$?

A. $f(x) = 1 - x^2$

B. $f(x) = x\left(x^2 - 1\right)$

C. $f(x) = x\left(1 - x^2\right)$

D. $f(x) = x^2\left(x^2 - 1\right)$

8 What is the vector projection of $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ in the direction of $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$?

A. $-\dfrac{20}{17}\underset{\sim}{i} - \dfrac{30}{17}\underset{\sim}{j}$

B. $-\dfrac{10}{13}\underset{\sim}{i} + \dfrac{40}{13}\underset{\sim}{j}$

C. $-\dfrac{20}{13}\underset{\sim}{i} - \dfrac{30}{13}\underset{\sim}{j}$

D. $-\dfrac{10}{17}\underset{\sim}{i} + \dfrac{40}{17}\underset{\sim}{j}$

9 The radius of a sphere, r, is increasing at the rate of 0.3 cm per second.
What is the rate of increase in the volume, V, in cm^3 per second, at the instant when the surface area is $100\pi\ \text{cm}^2$?

A. 10π

B. 12π

C. 25π

D. 30π

10 Which of the following is the range of the function $f(x) = \left|b\cos^{-1}(x) - a\right|$, where $a > 0$, $b > 0$ and $a < \dfrac{b\pi}{2}$?

A. $\left[-a,\ b\pi - a\right]$

B. $\left[0,\ b\pi - a\right]$

C. $\left[a,\ b\pi - a\right]$

D. $\left[0,\ a\right]$

End of Section I

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Solve $|2x-3| \le 1$. **2**

(b) Find $\displaystyle\int_0^{\frac{1}{2}} \frac{dy}{\sqrt{1-3y^2}}$. **2**

(c) Let α, β and γ be the roots of the equation $2x^3 - kx^2 - 4x + 12 = 0$.

(i) Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **2**

(ii) Given that two of its roots sum to zero, find the third root and hence find the value of k. **2**

(d) Using the substitution $t = \tan\dfrac{\theta}{2}$, or otherwise, show that **2**

$$\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} \quad \text{for all } \theta \ne k\pi,\ k \in \mathbb{Z}.$$

(e) Find the term independent of x in the expansion of $\left(3x^2 + \dfrac{2}{x}\right)^{12}$. **2**

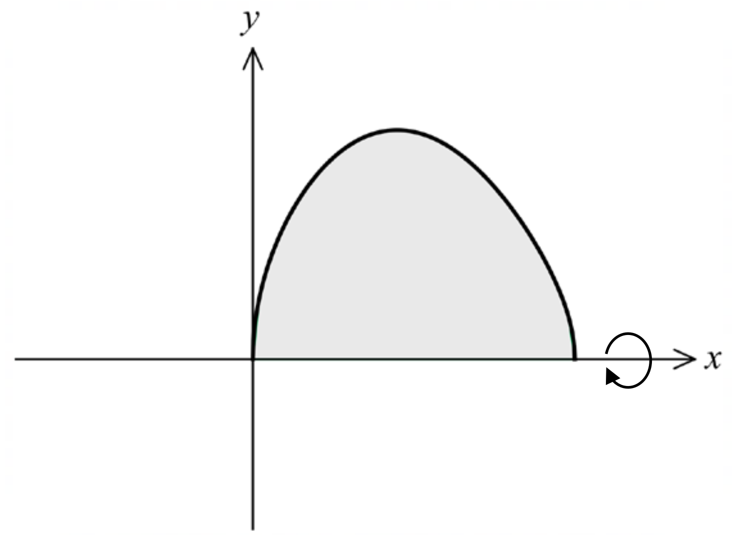
(f) Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all positive integers n. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Solve $\dfrac{x^2+6}{x} < 5.$ **3**

(b) By expressing $\cos x - \sqrt{3}\sin x$ in the form $A\cos(x+\alpha)$ where $A > 0,$ solve $\cos x - \sqrt{3}\sin x + 1 = 0$ for $0 \le x \le 2\pi.$ **4**

(c) A section of the graph of $y = \sqrt{\sin 3x \cos 2x}$ is shown in the diagram below. **4**

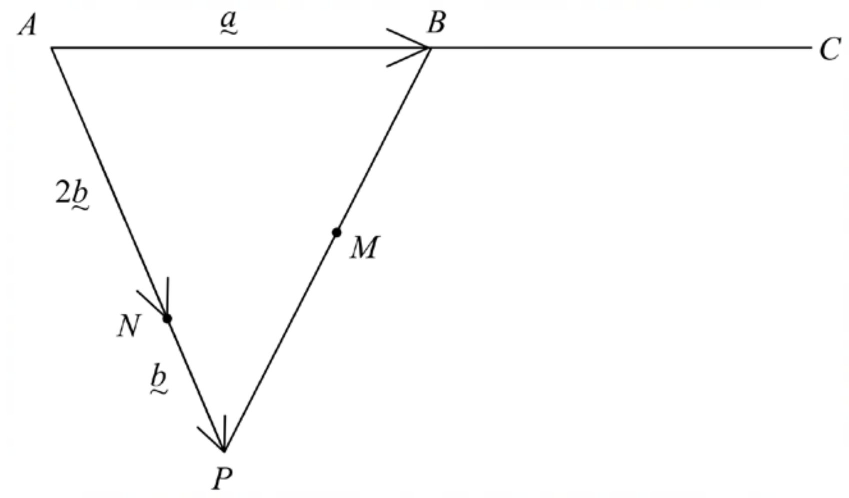


By first finding the smallest positive solution to $\sin 3x \cos 2x = 0,$ find the volume of the solid formed when the shaded region is rotated about the x-axis.

Question 12 continues on page 9

(d) In the diagram below APB is a triangle. N is a point on AP.

$\overrightarrow{AB} = \underset{\sim}{a}$ $\quad\quad$ $\overrightarrow{AN} = 2\underset{\sim}{b}$ $\quad\quad$ $\overrightarrow{NP} = \underset{\sim}{b}$



(i) Find the vector $\overrightarrow{PB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(ii) B is the midpoint of AC. M is the midpoint of PB. **3**
Show that NMC is a straight line.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) A netball team's record for the 2022 season was 16 wins and 4 losses. None of their games were drawn. Prove that the team must have won at least 4 games in a row somewhere during the season. **2**

(b) The letters of the word REORDER are arranged randomly in a line.

(i) Use a combinatorial argument to explain why **2**
$\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1}=\binom{7}{1}\binom{6}{1}\binom{5}{2}\binom{3}{3}$.

(ii) Hence, or otherwise, find the probability that a random rearrangement has all the consonants grouped together. **3**

(c) A pilot is performing at an air show. The position of her aeroplane at time t relative to a fixed origin O is given by $\underset{\sim}{r}(t)=\left(450-150\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{i}+\left(400-200\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a unit vector in a horizontal direction and $\underset{\sim}{j}$ is a unit vector vertically up. Displacement components are measured in metres and time t is measured in seconds where $t \geq 0$.

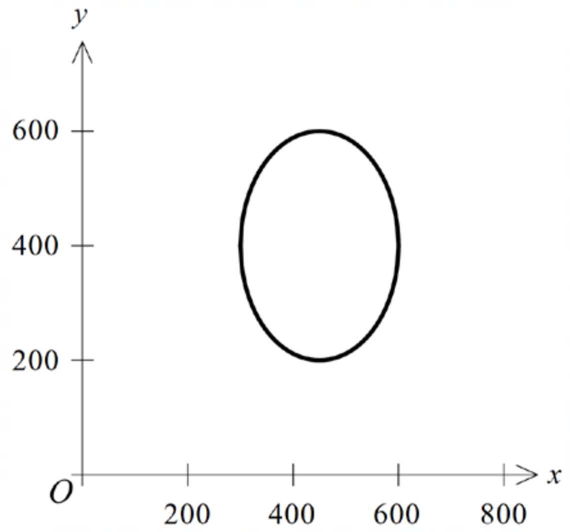
(i) Show that the cartesian equation of the path of the aeroplane is given by: **2**

$$\frac{(x-450)^2}{22500}+\frac{(y-400)^2}{40000}=1.$$

Question 13 continues on page 11

The path of the aeroplane is shown in the diagram below. At the same time that the pilot begins performing, a firework is fired from O with a velocity of 80 metres per second at an angle of inclination of θ. The position of the firework at time t relative to the fixed origin is given by $\underset{\sim}{s}(t) = (80t\cos\theta)\underset{\sim}{i} + (80t\sin\theta - 5t^2)\underset{\sim}{j}$.

(Do NOT prove this).



(ii) Find the value of θ given that the firework explodes when it reaches its maximum height of 160 m. **3**

(iii) By first finding a vector that represents the displacement of the aeroplane from the firework at time t, find how far the aeroplane is from the firework when it explodes. Give your answer to the nearest metre. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Use the substitution $x = \sin\theta$ to find $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$. **3**

(b) (i) Write $2\sin x \sin\left((2k+1)x\right)$ as the difference of two cosine functions. **1**

(ii) Prove by mathematical induction that for all integers $n \geq 1$, **3**

$$\sin x + \sin 3x + \sin 5x + \dots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x}.$$

(c) (i) The graph of $f(x) = -\frac{1}{x^2}$ is shown on the separate Response Sheet for Question 14 c (i). **2**

On the Response Sheet, use addition of ordinates to sketch the graph of $g(x) = x^2 - \frac{1}{x^2}$ for $y \in [-10, 10]$ clearly showing the location of the x-intercepts.
You do not need to find the x-coordinates at the endpoints of the range.

(ii) Show that $g(x)$ may be rearranged to give **2**

$$x^2 = \frac{y + \sqrt{y^2+4}}{2}.$$

A glass with a hollow stem, and with base at $y = -10$ is made by rotating the part of $g(x)$ where $x > 0$ and $y \in [-10, 10]$ about the y-axis to form a solid of revolution, where length units are in centimetres.

(iii) Write down a definite integral which, when evaluated, would give the volume of the glass. **1**

Do not attempt to evaluate this integral.

Question 14(c) continues on page 13

(iv) Liquid is poured into the glass at a rate of 1.5 cm^3 per second. **3**

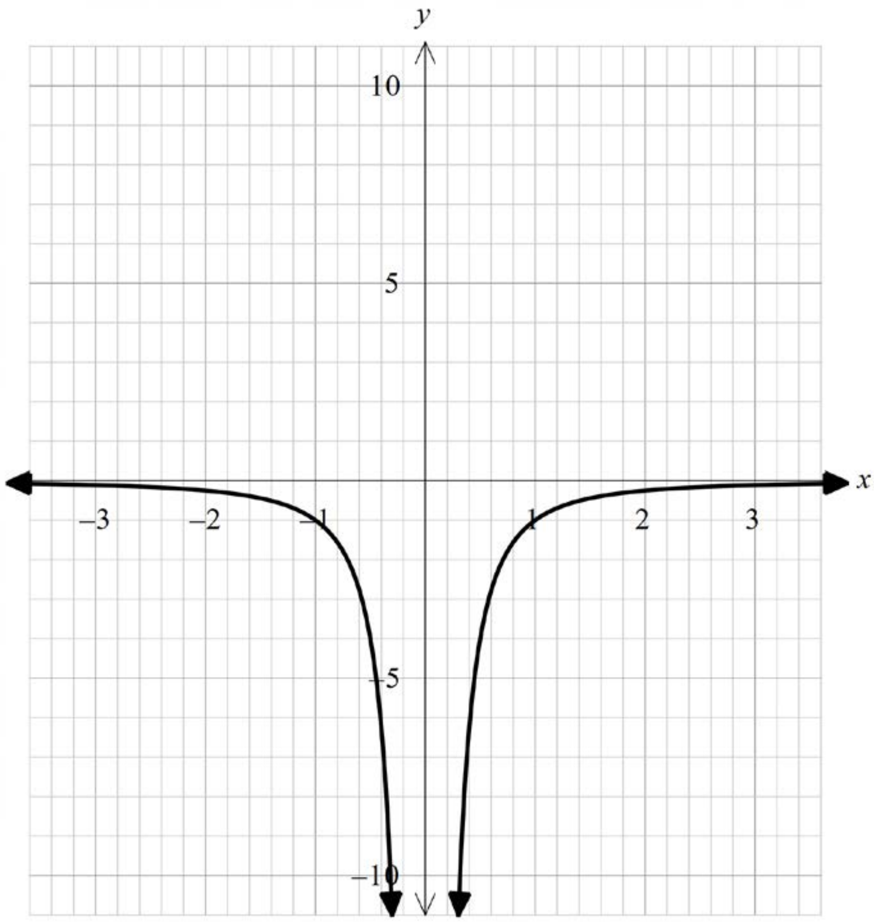
Find the rate at which the surface of the liquid is rising when it is 6 cm from the top of the glass.

End of paper

Student Number:________________

<u>Question 14 c) (i) – Response Sheet</u>

14 c) The graph of $f(x) = -\frac{1}{x^2}$ is shown below.



(i) On this Response Sheet provided, use addition of ordinates to sketch the graph of the $g(x) = x^2 - \frac{1}{x^2}$ for $y \in [-10, 10]$ clearly showing the location of the x-intercepts. **2**

You do not need to find the x-coordinates at the endpoints of the range.

Place this sheet inside your Question 14 answer booklet.

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2022 **HSC TRIAL EXAMINATION**

Mathematics Extension 1- Solutions

1 Let $P(x)=x^3-2ax^2+x-1$ where $a\in\mathbb{R}$. When $P(x)$ is divided by $x+2$, the remainder is 5. What is the value of a?

A. 2

B. $-\frac{7}{4}$

C. $\frac{1}{2}$

D. -2

$P(x) = x^3 - 2ax^2 + x - 1$

$P(-2) = (-2)^3 - 2a(-2)^2 + (-2) - 1$

$= -8 - 8a - 2 - 1$

$= -8a - 11 = 5$

$-8a = 16$

$\therefore a = -2$

2 The points A and B have coordinates $(-2,3)$ and $(2,-5)$ respectively.

Which of the following is the vector $\overrightarrow{AB}$?

A. $-2\underset{\sim}{j}$

B. $4\underset{\sim}{i}-8\underset{\sim}{j}$

C. $-4\underset{\sim}{i}+8\underset{\sim}{j}$

D. $2\underset{\sim}{j}$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$= \begin{pmatrix}2\\-5\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix}$

$= \begin{pmatrix}4\\-8\end{pmatrix}$

3 What is the angle between the vectors $\begin{pmatrix}-7\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$?

A. $\cos^{-1}(-0.8)$

B. $\cos^{-1}(-0.08)$

C. $\cos^{-1}(0.8)$

D. $\cos^{-1}(0.08)$

$\cos\theta = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$

$= \dfrac{\begin{pmatrix}-7\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}}{\sqrt{7^2+1^2}\sqrt{1^2+1^2}}$

$= \dfrac{-7-1}{\sqrt{50}\sqrt{2}}$

$= -\dfrac{8}{10}$

$\therefore \theta = \cos^{-1}(-0.8)$

4 Which of the following is the derivative of $\tan^{-1}(3x)$?

A. $3\tan^{-1}3x$

B. $\dfrac{3}{1+9x^2}$

C. $\dfrac{3}{1+3x^2}$

D. $3\sec^2 3x$

$\frac{d}{dx}\left[\tan^{-1}(3x)\right]$

$= \dfrac{3}{1+(3x)^2}$

$= \dfrac{3}{1+9x^2}$

5 What is the equation of the inverse of $f(x)=\dfrac{5+e^{2x}}{3}$?

A. $y=\dfrac{3}{5+e^{2x}}$

B. $y=e^{5-3x}$

C. $y=\dfrac{1}{2}\ln(3x-5)$

D. $y=\dfrac{1}{2}\ln(5-3x)$

f^{-1}: $x = \dfrac{5+e^{2y}}{3}$

$3x = 5+e^{2y}$

$e^{2y} = 3x-5$

$2y = \ln|3x-5|$

$y = \frac{1}{2}\ln|3x-5|$

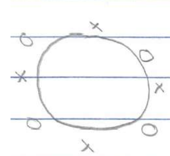
6 Four female and four male students are to be seated around a circular table.
In how many ways can this be done if the males and females must alternate?

A. $4!\times 4!$

B. $3!\times 4!$

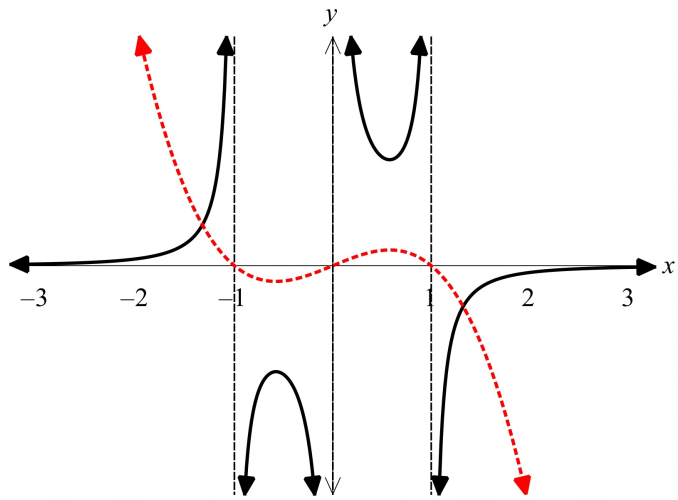
C. $3!\times 3!$

D. $2\times 3!\times 3!$



Arrange 4 females in 3! ways. then arrange males in the gaps in 4! ways.

7 The graph below shows $y = \dfrac{1}{f(x)}$.



Which of the following best represents the equation of $f(x)$?

A. $f(x) = 1 - x^2$

B. $f(x) = x\left(x^2 - 1\right)$

C. $f(x) = x\left(1 - x^2\right)$

D. $f(x) = x^2\left(x^2 - 1\right)$

$y = -x(x-1)(x+1)$

$= x(1-x^2)$

8 What is the vector projection of $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ in the direction of $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$?

A. $-\dfrac{20}{17}\underset{\sim}{i} - \dfrac{30}{17}\underset{\sim}{j}$

B. $-\dfrac{10}{13}\underset{\sim}{i} + \dfrac{40}{13}\underset{\sim}{j}$

C. $-\dfrac{20}{13}\underset{\sim}{i} - \dfrac{30}{13}\underset{\sim}{j}$

D. $-\dfrac{10}{17}\underset{\sim}{i} + \dfrac{40}{17}\underset{\sim}{j}$

$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$

$= \dfrac{2(1) + 3(-4)}{1+16}\begin{pmatrix}1\\-4\end{pmatrix}$

$= \dfrac{-10}{17}\begin{pmatrix}1\\-4\end{pmatrix}$

$= -\dfrac{10}{17}\underset{\sim}{i} + \dfrac{40}{17}\underset{\sim}{j}$

9 The radius of a sphere, r, is increasing at the rate of 0.3 cm per second.
What is the rate of increase in the volume, V, in cm^3 per second, at the instant when the surface area is $100\pi\ \text{cm}^2$?

A. 10π

B. 12π

C. 25π

D. 30π

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\quad SA = 4\pi r^2 = 100\pi$

Now $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = 100\pi \times 0.3 = 30\pi$

10 Which of the following is the range of the function $f(x) = \left|b\cos^{-1}(x) - a\right|$, where $a > 0$, $b > 0$ and $a < \frac{b\pi}{2}$?

A. $[-a, b\pi - a]$

B. $[0, b\pi - a]$

C. $[a, b\pi - a]$

D. $[0, a]$

$f(x) = |b\cos^{-1}(x) - a|$

$0 \le \cos^{-1}(x) \le \pi$

$0 \le b\cos^{-1}(x) \le b\pi$

$-a \le b\cos^{-1}(x) - a \le b\pi - a$

$0 \le |b\cos^{-1}(x) - a| \le b\pi - a$

$a < \frac{b\pi}{2}$

$2a < b\pi$

$0 < b\pi - 2a$

$\therefore b\pi - a > a > 0$

Question 11 (15 marks)

(a) Solve $|2x-3| \le 1$. **2**

$-1 \le 2x-3 \le 1$

$2 \le 2x \le 4$

$1 \le x \le 2$

(b) Find $\displaystyle\int_0^{\frac{1}{2}} \frac{dy}{\sqrt{1-3y^2}}$. **2**

$$\int_0^{1/2} \frac{dy}{\sqrt{1-3y^2}} = \frac{1}{\sqrt{3}} \int_0^{1/2} \frac{\sqrt{3}\,dy}{\sqrt{1-(\sqrt{3}y)^2}}$$

$$= \frac{1}{\sqrt{3}} \left[\sin^{-1}(\sqrt{3}y)\right]_0^{1/2}$$

$$= \frac{1}{\sqrt{3}} \left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(0)\right)$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

(c) Let α, β and γ be the roots of the equation $2x^3 - kx^2 - 4x + 12 = 0$.

(i) Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **2**

(ii) Given that two of its roots sum to zero, find the third root and hence find the value of k. **2**

(i) $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{-\frac{4}{2}}{-\frac{12}{2}}$$

$$= \frac{1}{3}.$$

ii) Let the roots be $\alpha, -\alpha, \beta$

$\frac{1}{\alpha} + \frac{1}{-\alpha} + \frac{1}{\beta} = \frac{1}{3}$ from part i)

$\frac{1}{\beta} = \frac{1}{3}$

$\beta = 3$

$\therefore \alpha - \alpha + 3 = \frac{k}{2}$ (sum of roots)

$3 = \frac{k}{2}$

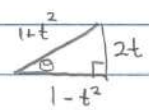
$k = 6$

(d) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, show that **2**

$$\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} \text{ for all } \theta \neq k\pi,\ k \in \mathbb{Z}.$$

LHS $= \cot\theta + \frac{1}{2}\tan\frac{\theta}{2}$

$= \frac{1-t^2}{2t} + \frac{1}{2}t$



$= \frac{1-t^2+t^2}{2t}$

$= \frac{1}{2t}$

$= \frac{1}{2} \times \frac{1}{t}$

$= \frac{1}{2} \times \cot\frac{\theta}{2}$

$=$ RHS

(e) Find the term independent of x in the expansion of $\left(3x^2 + \frac{2}{x}\right)^{12}$. **2**

$$T_k = \binom{12}{k}\left(3x^2\right)^{12-k}\left(2x^{-1}\right)^k$$

$$= \binom{12}{k} 3^{12-k} . x^{24-2k} . 2^k . x^{-k}$$

$$= \binom{12}{k} 3^{12-k} . 2^k x^{24-3k}$$

For term independent of x: $24-3k=0$

$k=8$.

$\therefore$ term independent of x $= \binom{12}{8} \times 3^4 \times 2^8$

$= 10264320$.

(f) Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all positive integers n. **3**

- Test for $n = 1$

  $1^3 + 2(1) = 3$ which is divisible by 3

  $\therefore$ The statement is true for $n = 1$

- Assume the statement is true for $n = k$

  i.e. $k^3 + 2k = 3P$ for integer P

- Prove for $n = k+1$

  i.e. $(k+1)^3 + 2(k+1) = 3Q$ $\quad Q \in \mathbb{Z}$

  $$\begin{aligned} LHS &= (k+1)^3 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= (k^3 + 2k) + 3(k^2 + k + 1) \\ &= 3P + 3(k^2 + k + 1) \quad \text{from assumption} \\ &= 3(P + k^2 + k + 1) \\ &= 3Q, \quad Q = P + k^2 + k + 1 \in \mathbb{Z} \end{aligned}$$

  $\therefore$ The statement is true for $n = k+1$, if it is true for $n = k$.

- By mathematical induction, it is true for all positive integers.

Question 12 (15 marks)

(a)	Solve $\dfrac{x^2+6}{x} < 5$.	**3**

a) $\dfrac{x^2+6}{x} < 5$

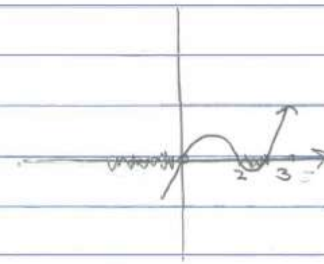
$(\times x^2)$ $\quad x(x^2+6) < 5x^2$

$x(x^2+6) - 5x^2 < 0$

$x(x^2-5x+6) < 0$

$x(x-3)(x-2) < 0$

$x < 0 \;,\; 2 < x < 3.$



(b)	By expressing $\cos x - \sqrt{3}\sin x$ in the form $A\cos(x+\alpha)$ where $A>0$, solve $\cos x - \sqrt{3}\sin x + 1 = 0$ for $0 \le x \le 2\pi$.	**4**

$\cos x - \sqrt{3}\sin x = A\cos(x+\alpha)$

$= A\cos x\cos\alpha - A\sin x\sin\alpha$

Equating coefficients: $\quad A\cos\alpha = 1 \quad (1)$

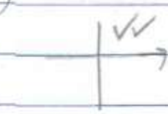
$A\sin\alpha = \sqrt{3} \quad (2)$

$(1)^2 + (2)^2$: $\quad A^2\cos^2\alpha + A^2\sin^2\alpha = 1+3$

$A^2 = 4$

$A = 2 \quad (A > 0)$

$(2) \div (1)$ $\quad \dfrac{A\sin\alpha}{A\cos\alpha} = \sqrt{3}$



$\tan\alpha = \sqrt{3}$

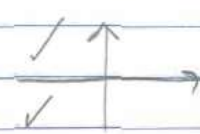
$\alpha = \pi/3$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x+\frac{\pi}{3}\right)$

$2\cos\left(x+\frac{\pi}{3}\right) + 1 = 0 \qquad 0 \le x \le 2\pi$

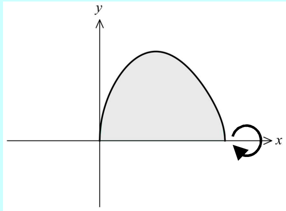
$\cos\left(x+\frac{\pi}{3}\right) = -\frac{1}{2} \qquad \frac{\pi}{3} \le x+\frac{\pi}{3} \le \frac{7\pi}{3}$

$x+\frac{\pi}{3} = \pi - \frac{\pi}{3},\; \pi + \frac{\pi}{3},$



$x = \frac{\pi}{3},\; \pi.$

(c) A section of the graph of $y = \sqrt{\sin 3x \cos 2x}$ is shown in the diagram below. **4**



By first finding the smallest positive solution to $\sin 3x \cos 2x = 0$, find the volume of the solid formed when the shaded region is rotated about the x-axis.

$\sin 3x \cos 2x = 0$

$\sin 3x = 0$, $\cos 2x = 0$

$3x = 0, \pi, 2\pi \ldots$ $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$ $x = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$

smallest positive solution $x = \frac{\pi}{4}$.

$V = \pi \int_0^{\pi/4} \left(\sqrt{\sin 3x \cos 2x}\right)^2 dx.$

$= \frac{\pi}{2} \int_0^{\pi/4} (\sin 5x + \sin x)\, dx.$

$= \frac{\pi}{2} \left[-\frac{\cos 5x}{5} - \cos x \right]_0^{\pi/4}$

$= \frac{\pi}{2} \left[\left(-\frac{\cos \frac{5\pi}{4}}{5} - \cos \frac{\pi}{4} \right) - \left(-\frac{\cos 0}{5} - \cos 0 \right) \right]$

$= \frac{\pi}{2} \left(\left(\frac{1}{5\sqrt{2}} - \frac{1}{\sqrt{2}} \right) - \left(-\frac{1}{5} - 1 \right) \right)$

$= \frac{\pi}{2} \left(\frac{1 - 5 + \sqrt{2} + 5\sqrt{2}}{5\sqrt{2}} \right)$

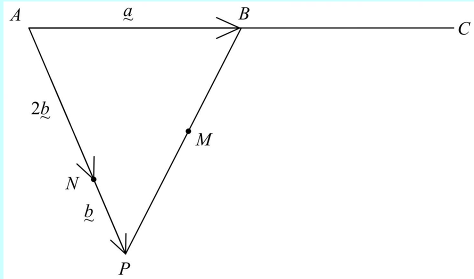
$= \frac{\pi}{10\sqrt{2}} \left(6\sqrt{2} - 4 \right)$

$= \frac{\pi}{5\sqrt{2}} \left(3\sqrt{2} - 2 \right)$

$= \frac{\pi}{10} \left(6 - 2\sqrt{2} \right) \; u^3$

(d) In the diagram below APB is a triangle. N is a point on AP.

$\overrightarrow{AB} = \underset{\sim}{a}$ $\quad\quad$ $\overrightarrow{AN} = 2\underset{\sim}{b}$ $\quad\quad$ $\overrightarrow{NP} = \underset{\sim}{b}$



(i) Find the vector $\overrightarrow{PB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(ii) B is the midpoint of AC. M is the midpoint of PB. **3**
Show that NMC is a straight line.

(i) $\overrightarrow{PB} = \overrightarrow{PA} + \overrightarrow{AB}$

$= -3\underset{\sim}{b} + \underset{\sim}{a}$

$= \underset{\sim}{a} - 3\underset{\sim}{b}$

(ii) $\overrightarrow{AC} = 2\underset{\sim}{a}$

$\overrightarrow{PM} = \frac{1}{2}\overrightarrow{PB}$

$= \frac{1}{2}(\underset{\sim}{a} - 3\underset{\sim}{b})$

$\overrightarrow{NM} = \overrightarrow{PM} - \overrightarrow{PN}$

$= \frac{1}{2}(\underset{\sim}{a} - 3\underset{\sim}{b}) - (-\underset{\sim}{b})$

$= \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$

$\overrightarrow{NC} = \overrightarrow{NA} + \overrightarrow{AC}$

$= -2\underset{\sim}{b} + 2\underset{\sim}{a}$

$= 2(\underset{\sim}{a} - \underset{\sim}{b})$

$\therefore \overrightarrow{NC} = 4\,\overrightarrow{NM}$

$\overrightarrow{NC}$ is a scalar multiple of $\overrightarrow{NM}$ and so NMC is a straight line.

Question 13 (15 marks)

(a)	A netball team's record for the 2022 season was 16 wins and 4 losses. None of their games were drawn. Prove that the team must have won at least 4 games in a row somewhere during the season.	**2**

There are 5 spots to place the wins between the 4 losses in the season

____ L ____ L ____ L ____ L ____

$\therefore$ There are 5 categories to place the 16 objects (wins).

$\frac{16}{5} = 3.2$

$\therefore$ There must be at least 4 wins in one of the categories. By the pigeonhole principal the team must have won 4 games in a row somewhere during the season.

(b)	The letters of the word REORDER are arranged randomly in a line.	
	(i) Use a combinatorial argument to explain why $\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \binom{7}{1}\binom{6}{1}\binom{5}{2}\binom{3}{3}$.	**2**
	(ii) Hence, or otherwise, find the probability that a random rearrangement has all the consonants grouped together.	**3**

R R R

E E   7 letters

O

D   _ _ _ _ _ _ _

To make an arrangement of these letters:

choose 3 spots for the R's in $\binom{7}{3}$ ways

then choose 2 spots for the E's from the remaining 4 spots

~~then~~ in $\binom{4}{2}$ ways.

then choose the spot for the O from the remaining 2 spots

in $\binom{2}{1}$ ways

and then finally place the D in $\binom{1}{1}$ ways.

$\therefore$ total # of arrangements $= \binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1}$

Alternately, first place the D in $\binom{7}{1}$ ways

then place the O in $\binom{6}{1}$ ways,

then place the E's in $\binom{5}{2}$ ways

and finally place the R's in $\binom{3}{3}$ ways.

$\therefore$ total # of arrangements $\binom{7}{1}\binom{6}{1}\binom{5}{2}\binom{3}{3}$

$\therefore$ $\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \binom{7}{1}\binom{6}{1}\binom{5}{2}\binom{3}{3}$

ii) Arrangements with consonants grouped together:

$\boxed{RRRD}$ OEE

4 groups

Arrange groups $\frac{4!}{2!}$ ways

and then arrange consonants in $\frac{4!}{3!}$ ways

$$P(\text{consonants grouped together}) = \frac{\frac{4!}{2!} \times \frac{4!}{3!}}{\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1}} \leftarrow \text{from (i)}$$

$$= \frac{4}{35}$$

(c) A pilot is performing at an air show. The position of her aeroplane at time t relative to a fixed origin O is given by $\underset{\sim}{r}(t) = \left(450 - 150\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{i} + \left(400 - 200\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j}$,

where $\underset{\sim}{i}$ is a unit vector in a horizontal direction and $\underset{\sim}{j}$ is a unit vector vertically up. Displacement components are measured in metres and time t is measured in seconds where $t \geq 0$.

(i) Show that the cartesian equation of the path of the aeroplane is given by: **2**

$$\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = 1.$$

c) (i) $\underset{\sim}{r}(t) = \begin{bmatrix} 450 - 150\sin\left(\frac{\pi t}{6}\right) \\ 400 - 200\cos\left(\frac{\pi t}{6}\right) \end{bmatrix}$

$x = 450 - 150\sin\left(\frac{\pi t}{6}\right) \Rightarrow \frac{x-450}{-150} = \sin\left(\frac{\pi t}{6}\right)$ ①

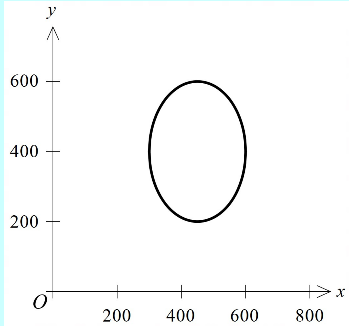
$y = 400 - 200\cos\left(\frac{\pi t}{6}\right) \Rightarrow \frac{y-400}{-200} = \cos\left(\frac{\pi t}{6}\right)$ ②

①^2 + ②^2 $\quad \frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = \sin^2\left(\frac{\pi t}{6}\right) + \cos^2\left(\frac{\pi t}{6}\right)$

$\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40\,000} = 1.$

The path of the aeroplane is shown in the diagram below. At the same time that the pilot begins performing, a firework is fired from O with a velocity of 80 metres per second at an angle of inclination of θ. The position of the firework at time t relative to the fixed origin is given by $\underset{\sim}{s}(t) = (80t\cos\theta)\underset{\sim}{i} + (80t\sin\theta - 5t^2)\underset{\sim}{j}$.

(Do NOT prove this).



(ii) Find the value of θ given that the firework explodes when it reaches its maximum height of 160 m. **3**

(iii) By first finding a vector that represents the displacement of the aeroplane from the firework at time t, find how far the aeroplane is from the firework when it explodes. Give your answer to the nearest metre. **3**

ii) $\underset{\sim}{s}(t) = \begin{bmatrix} 80t\cos\theta \\ 80t\sin\theta - 5t^2 \end{bmatrix}$

$y = 80t\sin\theta - 5t^2$

$\dot{y} = 80\sin\theta - 10t$

When $\dot{y} = 0$, $80\sin\theta - 10t = 0$

$t = 8\sin\theta$

when $t = 8\sin\theta$, $y = 160$

$80(8\sin\theta)\sin\theta - 5(8\sin\theta)^2 = 160$

$640\sin^2\theta - 320\sin^2\theta = 160$

$320\sin^2\theta = 160$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$ $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$

iii) $\vec{SR} = \vec{OR} - \vec{OS}$

$$= \begin{bmatrix} 450 - 150\sin\left(\frac{\pi t}{6}\right) \\ 400 - 200\cos\left(\frac{\pi t}{6}\right) \end{bmatrix} - \begin{bmatrix} 80t\cos\theta \\ 80t\sin\theta - 5t^2 \end{bmatrix}$$

$\theta = \frac{\pi}{4} \Rightarrow \sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$. $\quad t = 8\sin\theta$.

$= 8/\sqrt{2}$

$$\vec{SR} = \begin{bmatrix} 450 - 150\sin\left(\frac{\pi}{6} \times \frac{8}{\sqrt{2}}\right) - 80\left(\frac{8}{\sqrt{2}}\right) \times \frac{1}{\sqrt{2}} \\ 400 - 200\cos\left(\frac{\pi}{6} \times \frac{8}{\sqrt{2}}\right) - 80\left(\frac{8}{\sqrt{2}}\right) \times \frac{1}{\sqrt{2}} + 5\left(\frac{8}{\sqrt{2}}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 450 - 150\sin\left(\frac{2\sqrt{2}\pi}{3}\right) - 320 \\ 400 - 200\cos\left(\frac{2\sqrt{2}\pi}{3}\right) - 320 + 160 \end{bmatrix}$$

$$= \begin{bmatrix} 130 - 150\sin\left(\frac{2\sqrt{2}\pi}{3}\right) \\ 240 - 200\cos\left(\frac{2\sqrt{2}\pi}{3}\right) \end{bmatrix}$$

$$|\vec{SR}| = \sqrt{\left(130 - 150\sin\left(\frac{2\sqrt{2}\pi}{3}\right)\right)^2 + \left(240 - 200\cos\left(\frac{2\sqrt{2}\pi}{3}\right)\right)^2}$$

$= \sqrt{201426.25\ldots}$

$\doteqdot 448.8$ m.

$\doteqdot 449$ m (nearest metre)

Question 14 (15 marks)

(a)	Use the substitution $x = \sin\theta$ to find $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$.	**3**

$$\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$$

Let $x = \sin\theta \Rightarrow \theta = \sin^{-1}(x)$

$dx = \cos\theta\, d\theta$.

when $x = 0$, $\theta = 0$

when $x = \frac{1}{2}$, $\theta = \frac{\pi}{6}$.

$$= \int_0^{\pi/6} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta.$$

$$= \int_0^{\pi/6} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta.$$

$$= \int_0^{\pi/6} \frac{\sin^2\theta \cancel{\cos\theta}}{\cancel{\cos\theta}}\, d\theta. \qquad \cos\theta > 0$$

$$= \int_0^{\pi/6} \frac{1}{2}(1-\cos 2\theta)\, d\theta.$$

$$= \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$$

$$= \left[\left(\frac{\pi}{12} - \frac{1}{4}\sin\left(\frac{\pi}{3}\right)\right) - (0 - 0)\right]$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8} = \frac{2\pi - 3\sqrt{3}}{24}$$

(b)	(i)	Write $2\sin x \sin\left(\left(2k+1\right)x\right)$ as the difference of two cosine functions.	**1**
	(ii)	Prove by mathematical induction that for all integers $n \geq 1$, $$\sin x + \sin 3x + \sin 5x + ... + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x}.$$	**3**

i) $2\sin x \sin\left(\left(2k+1\right)x\right) = \cos\left(x - (2k+1)x\right) - \cos\left(x + (2k+1)x\right)$

$= \cos\left(-2kx\right) - \cos\left(2(k+1)x\right)$

$= \cos\left(2kx\right) - \cos\left(2(k+1)x\right)$ (cos is even fn)

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x}$$

• Test for $n=1$

$$LHS = \sin(2(1)-1)x = \sin x$$

$$RHS = \frac{1-\cos 2(1)x}{2\sin x} = \frac{1-\cos 2x}{2\sin x} = \frac{2\sin^2 x}{2\sin x} = \sin x$$

$\therefore$ True for $n=1$

• Assume true for $n=k$

i.e. $\sin x + \sin 3x + \sin 5x + \cdots + \sin(2k-1)x = \dfrac{1-\cos 2kx}{2\sin x}$

• Prove true for $n=k+1$

i.e. $\sin x + \sin 3x + \sin 5x + \cdots + \sin(2(k+1)-1)x = \dfrac{1-\cos 2(k+1)x}{2\sin x}$

$$LHS = \sin x + \sin 3x + \sin 5x + \cdots + \sin(2k-1)x + \sin(2(k+1)-1)x$$

$$= \frac{1-\cos 2kx}{2\sin x} + \sin(2(k+1)-1)x \quad \text{using assumption}$$

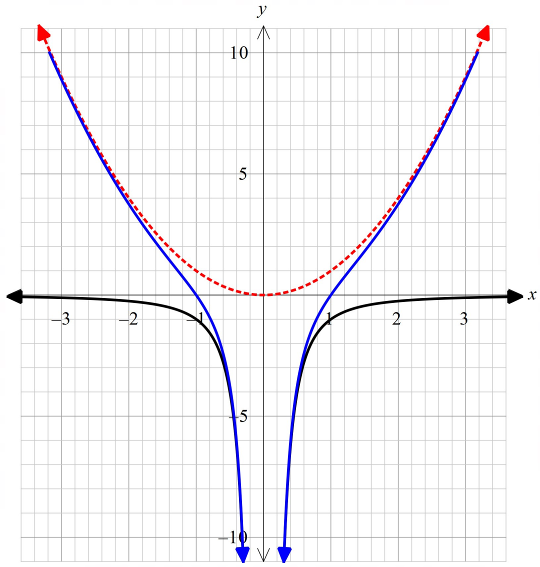
$$= \frac{1-\cos 2kx + 2\sin x \sin(2k+1)x}{2\sin x}$$

$$= \frac{1-\cancel{\cos 2kx} + \cancel{\cos(2kx)} - \cos 2(k+1)x}{2\sin x} \quad \text{from part i}$$

$$= \frac{1-\cos 2(k+1)x}{2\sin x} = RHS$$

$\therefore$ By mathematical induction it is true for all integers $n \geq 1$

(c)	(i)	The graph of $f(x) = -\dfrac{1}{x^2}$ is shown below. Use addition of ordinates to sketch the graph of $g(x) = x^2 - \dfrac{1}{x^2}$ for $y \in [-10, 10]$ clearly showing the location of the x-intercepts. **You do not need to find the x-coordinates at the endpoints of the range.**	**2**



(ii)	Show that $g(x)$ may be rearranged to give $x^2 = \dfrac{y + \sqrt{y^2 + 4}}{2}$.	**2**

$g(x) = x^2 - \frac{1}{x^2}$

$y = x^2 - \frac{1}{x^2}$

$yx^2 = x^4 - 1 \Rightarrow x^4 - x^2 y - 1 = 0$

$x^2 = \dfrac{y \pm \sqrt{(-y)^2 - 4(1)(-1)}}{2}$

$= \dfrac{y \pm \sqrt{y^2 + 4}}{2}$

$\therefore x^2 = \dfrac{y + \sqrt{y^2 + 4}}{2} \quad (x^2 > 0)$

A glass with a hollow stem, and with base at $y = -10$ is made by rotating the part of $g(x)$ where $x > 0$ and $y \in [-10, 10]$ about the y-axis to form a solid of revolution, where length units are in centimetres.

(iii) Write down a definite integral which, when evaluated, would give the volume of the glass. **1**

(iv) Liquid is poured into the glass at a rate of $1.5\ \text{cm}^3$ per second. **3**
Find the rate at which the surface of the liquid is rising when it is 6 cm from the top of the glass.

iii) $V = \pi \int_{-10}^{10} \frac{y + \sqrt{y^2+4}}{2}\, dy.$

iv) $\frac{dV}{dt} = 1.5.$ Find $\frac{dy}{dt}$ when $y = 4$ (top of glass is $y=10$)

$V(y) = \int_{-10}^{y} \pi \frac{(y + \sqrt{y^2+4})}{2}\, dy = \int_{-10}^{y} \frac{dV}{dy}\, dy$

$\frac{dV}{dt} = \frac{dV}{dy} \times \frac{dy}{dt}$

$1.5 = \pi \frac{(y + \sqrt{y^2+4})}{2} \times \frac{dy}{dt}$

At $y = 4$

$1.5 = \pi \left(\frac{4 + \sqrt{16+4}}{2}\right) \times \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{1.5 \times 2}{\pi(4 + \sqrt{20})} = 0.1127\ldots \doteq 0.1$

Rising at a rate of 0.1 cm/s